

PRESSURE DECONVOLUTION IN ELASTIC CONTACTS

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Abstract: When applying superposition principle to elastic contact problems, many of the arising equations become convolution type products. The need for increased computational efficiency has welcomed spectral methods into contact numerical modeling. The most notorious implementation is the computation of displacement field due to a digitized pressure distribution. The inverse problem, namely that of the pressure distribution which induce a known displacement field, may equally be formulated. It corresponds to convolution inverse operation, which is deconvolution. Problem periodization due to frequency domain transfer is expected to pollute the deconvoluted pressure, especially near the boundaries of the computational domain. Numerical simulations are performed to check method viability, demonstrating that the periodicity error can be controlled by domain extension.

1. INTRODUCTION

As equations arising in most models of contact mechanics lack analytical resolution, numerical methods are increasingly used due to their robustness. In the numerical formulation, fine discretization of investigated domains yields systems with large numbers of equations, which cannot be solved efficiently using conventional numerical methods. Spectral methods are considered nowadays mandatory tools when dealing with contact mechanics numerical simulations. As many of arising equations are in fact convolution or correlation products, transfer to frequency domain, where, according to convolution theorem, multi summation is substituted by element-wise multiplication, appear as the natural choice for decreasing the computational effort. However, this short cut does not come without a price, which is the error introduced by problem periodization.

Ju and Farris, [3], were the first to use fast Fourier transform to evaluate linear convolutions in contact problems. Nogi and Kato, [9], derived influence coefficients from frequency response function. They stated that a two times domain extension is enough to reduce periodicity error. Liu, Wang and Liu, [6], made a detailed analysis of error sources when computing convolutions using spectral methods. They classified existing methods, and proposed a new algorithm, capable of completely avoiding the periodicity error, called Discrete Convolution Fast Fourier Transform (DCFFT). Later on, Liu and Wang, [5], used DCFFT to compute stress fields in elastic layered half-space. They advanced a method to derive influence coefficients from frequency response function, when the corresponding Green function is not known. The DCFFT algorithm was later used by many authors [1,10-13] in simulations of elastic and elastic-plastic contact, with rough or nominal contact geometry.

In elastic contact problem, displacement field due to an arbitrarily pressure is expressed as a convolution product, which is performed repeatedly in an iterative manner, until pressure distribution and contact area converge. With this formulation, pressure is known, and displacement field must be computed. Another variation can be equally formulated, when displacement field is known, and pressure distribution is the unknown. If the former formulation corresponds to a convolution, the latter is associated with convolution inverse operation, which is deconvolution.

Deconvolution was used to assess pressure distribution by Liu et al., [7]. They stated that, according to span rule, extension of computational domain with a factor of three is enough to completely avoid any periodicity error.

An algorithm to derive pressure distribution by deconvolution is advanced in this paper. Periodicity error is dealt with via domain extension. Numerical simulations for circular Hertz contact suggest that the algorithm is feasible. Therefore, it can be further integrated in a program for solving elastic contact problem, as the one proposed by Kawabata and Nakamura, [4], requiring inversion of influence coefficients matrix.

2. INFLUENCE COEFFICIENTS COMPUTATION

When a unit point force acts normally on the surface of an elastic isotropic half-space, the force-displacement relation follows Boussinesq fundamental solution. If a Cartesian coordinate system with its origin attached to the point of force action is used, the force-displacement Green function, assessing the response of the system (the elastic half-space) to a unit impulse (the unit point force), can be written:

$$G(x_1, x_2) = u_3(x_1, x_2, 0) = \frac{\eta}{\pi \sqrt{x_1^2 + x_2^2}}. \quad (1)$$

with η the elastic constant of the contact. Although G has a singular point in origin, it is integrable everywhere. If an arbitrarily distributed load is assumed to act on a region Γ_C of half-space boundary, surface normal displacement at every point (x_1, x_2) results from superposition principle:

$$u_3(x_1, x_2) = \iint_{\Gamma_C} p_3(x'_1, x'_2) G(x_1 - x'_1, x_2 - x'_2) dx'_1 dx'_2. \quad (2)$$

Closed form integration in Eq. (2) is not available except for a few cases. Therefore, numerical approach is preferred. In the digitized formulation of the contact problem, all distributions are assumed as piecewise constant on the grids of a rectangular mesh established on the half-space boundary, covering a domain including Γ_C .

Normal displacement due to a uniform pressure acting on a rectangular domain yields from integration of corresponding Green function (1) with respect to directions of $\bar{x}_1$ and $\bar{x}_2$ over a rectangular domain. The first solution was presented by Love, [8]. If the set of grids is denoted by A_p and two integers (k, ℓ) are used to index elementary cells, in the numerical formulation, Eq. (2) reduces to:

$$u_3(i, j) = \sum_{(k, \ell) \in A_p} \left(p(k, \ell) \int_{x_2(\ell) - \Delta_2/2}^{x_2(\ell) + \Delta_2/2} \int_{x_1(k) - \Delta_1/2}^{x_1(k) + \Delta_1/2} G(x_1(i) - x'_1, x_2(j) - x'_2) dx'_1 dx'_2 \right), \quad (3)$$

with Δ_1 and Δ_2 grid steps in the directions of $\bar{x}_1$ and $\bar{x}_2$ respectively. The double integral in (3) is also known as the influence coefficient:

$$K(i - k, j - \ell) = \int_{x_2(\ell) - \Delta_2/2}^{x_2(\ell) + \Delta_2/2} \int_{x_1(k) - \Delta_1/2}^{x_1(k) + \Delta_1/2} G(x_1(i) - x'_1, x_2(j) - x'_2) dx'_1 dx'_2, \quad (4)$$

and represents the displacement at cell (i, j) generated by a unit uniform pressure acting on the (k, ℓ) elementary cell. The following closed form solution was advanced, [2]:

$$K(i, j) = \frac{\eta}{\pi} \left(\begin{aligned} &f(x_1(i) + \Delta_1/2, x_2(j) + \Delta_2/2) + f(x_1(i) - \Delta_1/2, x_2(j) - \Delta_2/2) \dots \\ &-f(x_1(i) + \Delta_1/2, x_2(j) - \Delta_2/2) - f(x_1(i) - \Delta_1/2, x_2(j) + \Delta_2/2) \end{aligned} \right), \quad (5)$$

with:

$$f(x_1, x_2) = x_1 \ln \left(x_2 + \sqrt{x_1^2 + x_2^2} \right) + x_2 \ln \left(x_1 + \sqrt{x_1^2 + x_2^2} \right). \quad (6)$$

According to superposition principle, surface normal displacement field can be expressed as a multisummation:

$$u_3(i, j) = \sum_{(k, \ell) \in A_p} K(i - k, j - \ell) p(k, \ell) = \sum_{k=1}^{N_1} \sum_{\ell=1}^{N_2} K(i - k, j - \ell) p(k, \ell), \quad (7)$$

which can be performed numerically for all types of pressure distributions and shapes of A_p . However, in order to keep the discretization error to an acceptable level, a fine grid must be imposed, which results in increased computational requirements. To speed up the computation, Liu, Wang and Liu, [6], developed the DCFET technique, which relies on the convolutive nature of product arising in Eq. (7).

3. PRESSURE DECONVOLUTION ALGORITHM

Eq. (7) is a convolution in time/space domain with respect to directions $\bar{x}_1$ and $\bar{x}_2$:

$$\mathbf{u} = \mathbf{K} \otimes \mathbf{p}, \quad (8)$$

where " $\otimes$ " operator is used to denote discrete cyclic convolution. Displacement field computation by direct multiplication in Eq. (7) would require $O(N^2)$ operations. By applying discrete cyclic convolution theorem, the same product can be computed in only $O(N \log N)$ in the frequency domain, according to DCFET applied convolution theorem.

As operation in Eq. (8) is a convolution, when considering this relation as an equation in $\mathbf{p}$, the unknown can be derived with convolution inverse operation, which is deconvolution:

$$\mathbf{p} = \mathbf{K}^{-1} \cdot \mathbf{u}. \quad (9)$$

However, pressure computation from Eq. (8) is not efficient in the time/space domain, as requires inversion of influence coefficients matrix, namely $\mathbf{K}^{-1}$. This operation can be performed in $O(N^3)$, thus limiting the number of grids to be imposed. On the other hand, as convolution in time/space domain is transformed in element-wise multiplication in the frequency domain, influence coefficients matrix inversion requires only $O(N)$ operations of complex division (" $\div$ ") as spectral series. Therefore, solution to Eq. (9) can be obtained more efficiently in the frequency domain, by deconvolution.

$$\hat{\mathbf{p}} = \hat{\mathbf{u}} \div \hat{\mathbf{K}}. \quad (10)$$

A hat (" $\hat{g}$ ") is used here to denote the spectral counterpart of any real series g . However, when applying discrete Fourier transform to any real series, one tacitly assumes that the corresponding distribution is periodical, with the period equal to the window for which information is mapped. Consequently, the contact problem, which is not usually periodical, (rough contact can be treated as periodical, by selecting a representative domain, as performed in [1]), is treated as periodical. Consequently, the result of the computation is determined not only by the investigated window, but also by the neighboring ones. This pollution of output is also referred to as periodicity error, which is significant especially at the boundaries of the investigated domain. In order to minimize periodicity error, one way is to widen the window by zero-padding, thus limiting the contribution of the neighboring periods.

The algorithm proposed in this paper for pressure deconvolution in elastic contacts is based on the DCFFT technique advanced by Liu, Wang and Liu, [6]. It involves computation on an extended domain, also referred to as the computational domain, which is χ times wider (on every direction) than the original one, referred to as the target domain.

The algorithm consists of the following steps:

1. Acquire the input: elastic parameters of the contacting bodies (Young modulus, E , Poisson's ratio, ν), grid parameters for the target domain (grid sides, L_1, L_2 , number of grids, N_1, N_2), and the ratio χ between target and computational domain.
2. Acquire mapped displacement field on the computational domain, $u(i, j)$, $i = \overline{1, \chi N_1}$, $j = \overline{1, \chi N_2}$. This can be achieved by laser profilometry, if one of the contacting bodies is optically transparent, or from existing closed form solutions.
3. Compute influence coefficients matrix on the computational domain $K(i, j)$, $i = \overline{1, \chi N_1}$, $j = \overline{1, \chi N_2}$.
4. Rearrange the terms of $\mathbf{K}$ in wrap-around order. The terms corresponding to negative arguments are transferred to the positive side, as depicted in Fig. 1, for $\chi N_1 = \chi N_2 = N$.

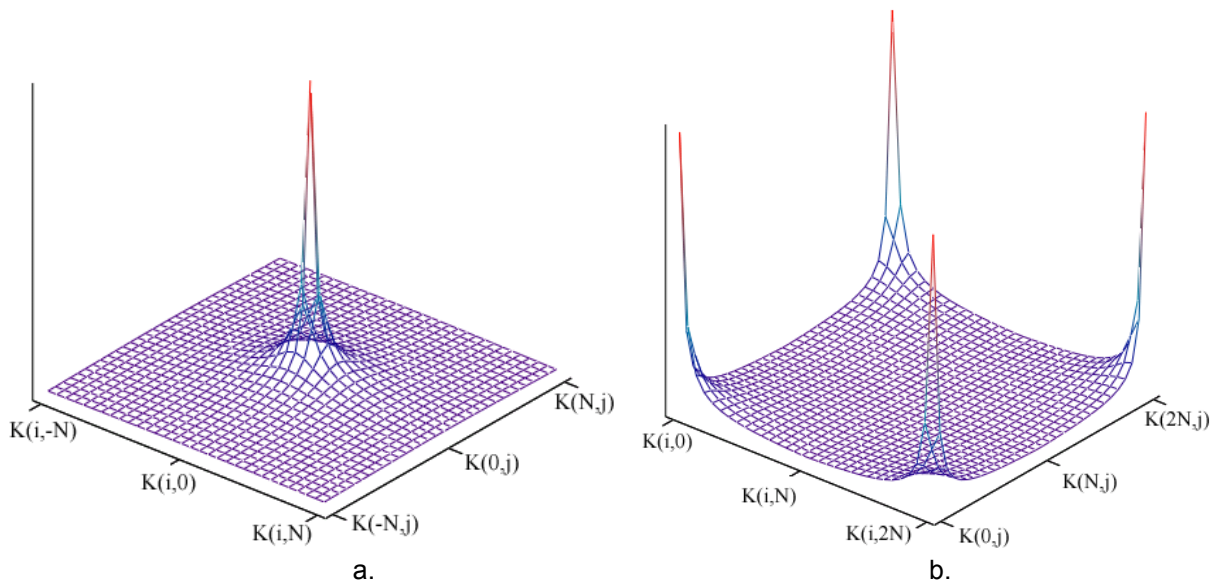


Figure 1. Influence coefficients: a. initial position, b. rearranged in wrap-around order

As function described by Eq. (5) is even, namely $K(-i, -j) = K(i, -j) = K(-i, j) = K(i, j)$, it is redundant to compute coefficients $\mathbf{K}$ in all quadrants.

5. Compute the Fourier transforms of $\mathbf{K}$ and $\mathbf{u}$ by means of a fast Fourier transform two-dimensional algorithm, thus obtaining $\hat{\mathbf{K}}$ and $\hat{\mathbf{u}}$, both of size $\chi N_1 \times \chi N_2$ complex numbers.

6. Compute a temporary spectral series by element-wise complex division: $\hat{p}(i, j) = \hat{u}(i, j) \div \hat{K}(i, j)$, $i = \overline{1, \chi N_1}$, $j = \overline{1, \chi N_2}$.

7. Transfer pressure back to time/space domain by means of an inverse fast Fourier transform algorithm, thus obtaining the complex series $p(i, j)$, $i = \overline{1, \chi N_1}$, $j = \overline{1, \chi N_2}$.

8. Retain the real part of the values corresponding to target domain, thus obtaining the output: $p(i, j)$, $i = \overline{1, N_1}$, $j = \overline{1, N_2}$.

In the following section, method accuracy is investigated for different values of χ . However, it should be noted that most of the computational resources are necessary to compute two-dimensional fast Fourier transforms. Existing algorithms are most efficient when the numbers of terms in the input is a power of two. Even when this condition is not met, zero-padding to the first power of two is performed internally. Consequently, an efficient approach requires χ to be a power of two as well.

RESULTS AND DISCUSSIONS

To validate the proposed algorithm, pressure is deconvoluted in Hertz circular contact. Various ratios between sides of the computational and of the target domain, both squares, are imposed, $\chi = L/L_0$. For this type of contact, a closed form expression for surface normal displacement was advanced by Johnson, [2]:

$$u(r) = \begin{cases} \pi \eta p_H \frac{2a_H^2 - r^2}{2a_H}, & |r| < a_H; \\ \eta \frac{p_H}{a_H} \left[(2a_H^2 - r^2) \sin^{-1}(a_H/r) + ra_H \sqrt{1 - (a_H/r)^2} \right], & |r| \geq a_H, \end{cases} \quad (11)$$

where $1/\eta$ is the effective elastic modulus of the contact, p_H hertzian pressure and a_H Hertz contact radius. Relation (11) can be used to generate the digitized displacement field $\mathbf{u}$ for any value of χ .

The deconvoluted pressure is then compared to the analytical curve for this type of contact, for three values of χ , namely $\chi = 2$, $\chi = 4$, and $\chi = 8$. The pressure profiles are depicted in Figs 2 and 3.

These results suggest that pressure distribution in elastic contacts can be assessed by deconvolution with an acceptable accuracy if the computational domain is four times or, for more precise results, eight times (on every direction) the target domain. When this ratio is only $\chi = 2$, pressure is underestimated, and tensile tractions are predicted outside contact area.

In all cases, the most important errors occur near the boundary of the computational domain, due to influence of neighboring periods. As these polluted values are discarded, they have no influence on algorithm output.

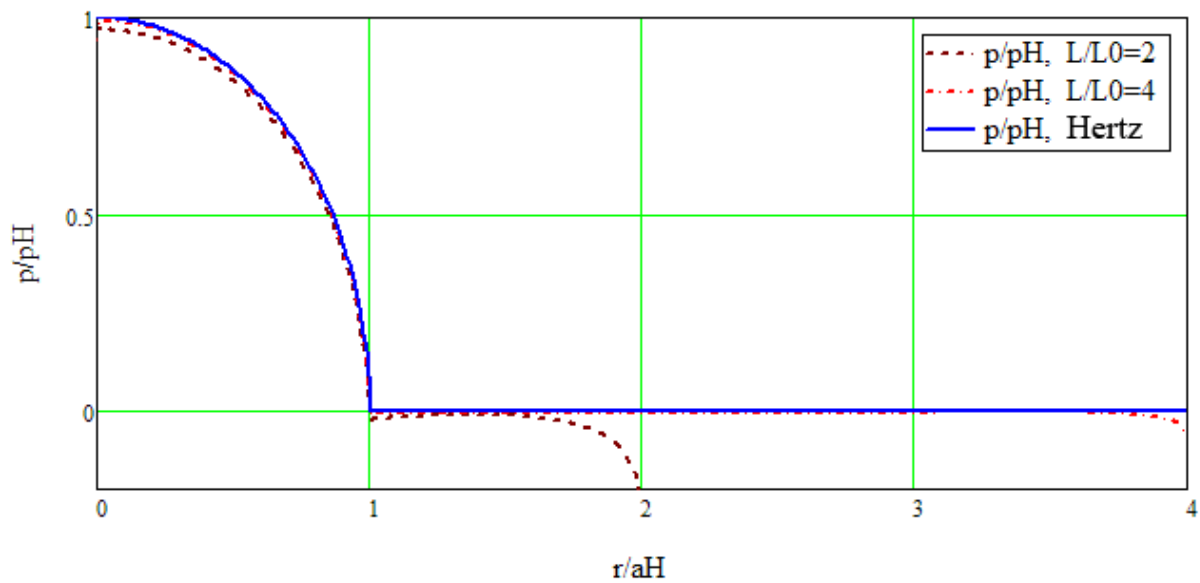


Figure 2. Pressure profiles for Hertz circular contact, $\chi = 2$ and $\chi = 4$

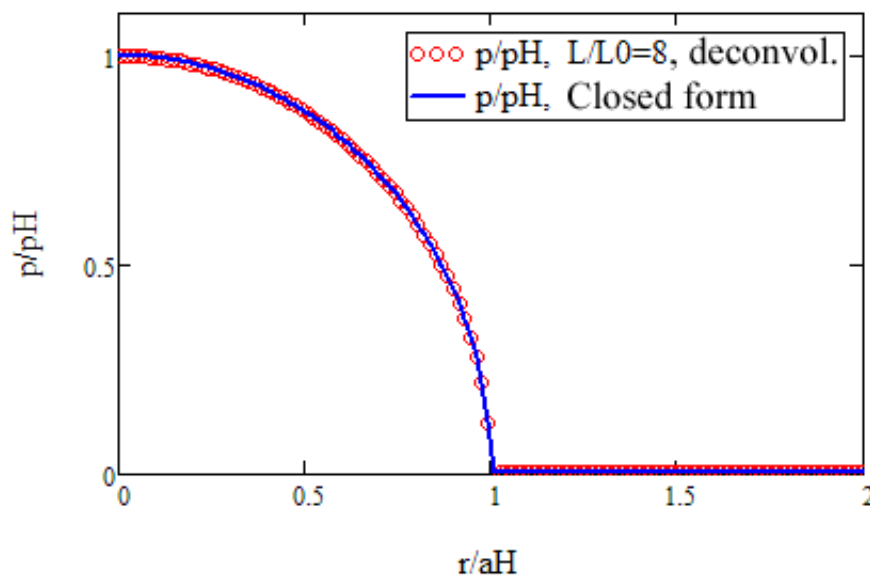


Figure 3. Pressure profiles for Hertz circular contact, $\chi = 8$

CONCLUSIONS

An efficient algorithm to compute pressure distribution in an elastic contact for which surface displacement field is known was advanced and verified in this paper. The method relies on the convolutive nature of equation relating pressure and surface deflections. Influence coefficients matrix inversion is time-consuming in the time/space domain, being of order $O(N^3)$. Therefore, pressure is derived by deconvolution in the frequency domain, which requires only $O(N \log N)$ operations.

Transfer of time/space series to spectral domain implies problem periodization, affecting the output through periodicity error. The contribution of spurious neighboring periods can be minimized by domain extension.

Algorithm for pressure deconvolution is verified for circular Hertz contact, using closed form expressions for surface displacement derived by Johnson. Numerical simulation

suggests that accurate pressure distributions can be obtained when the computational domain is eight times wider, on every direction, than the one for which pressure is sought.

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